**Lattices**

**Definition:** A lattice is a [Partial order](http://lara.epfl.ch/web2010/partial_order) in which every two-element set has a least upper bound and a greatest lower bound.

**Lemma:** In a lattice every non-empty finite set has a lub ( $\sqcup$) and glb ( $\sqcap$).

**Proof:** is by induction!  
Case where the set S has three elements x,y and z:  
Let $a=(x \sqcup y) \sqcup z$.   
By definition of $\sqcup$we have $z \sqsubseteq a$and $ x \sqcup y \sqsubseteq a $.  
The we have again by definition of $\sqcup$, $x \sqsubseteq x \sqcup y$and $y \sqsubseteq x \sqcup y$. Thus by transitivity we have $x \sqsubseteq a$and $y \sqsubseteq a$.  
Thus we have $S \sqsubseteq a$and a is an upper bound.  
Now suppose that there exists $a^\prime$such that $S \sqsubseteq a^\prime$. We want $a \sqsubseteq a^\prime$(a least upper bound):  
We have $x \sqsubseteq a^\prime $and $x \sqsubseteq a^\prime $, thus $x \sqcup y \sqsubseteq a^\prime$. But $z \sqsubseteq a^\prime $, thus $((x \sqcup y) \sqcup z) \sqsubseteq a^\prime $.  
Thus $a$is the lub of our 3 elements set.

**Lemma:** Every [linear order](http://lara.epfl.ch/web2010/sav08:homework08#problem_2) is a lattice.

If a lattice has least and greatest element, then every finite set (including empty set) has a lub and glb.

This does not imply there are lub and glb for infinite sets.

**Example:** In the oder $([0,1),\le)$with standard ordering on reals is a lattice, the entire set has no lub. The set of all rationals of interval $[0,10]$is a lattice, but the set $\{ x \mid 0 \le x \land x^2 < 2 \}$has no lub.

**Definition:** A **complete** lattice is a lattice where every set $S$of elemenbts has lub, denoted $\sqcup S$, and glb, denoted $\sqcap S$(this implies that there is top and bottom as $\sqcup \emptyset = \bot$and $\sqcap \emptyset = \top$. This is because every element is an upper bound and a lower bound of $\emptyset$: $\forall x. \forall y \in \emptyset. y \sqsubseteq x$is valid, as well as $\forall x. \forall y \in \emptyset. y \sqsupseteq x$).

Note: if you know that you have least upper bounds for all sets, it follows that you also have greatest lower bounds.

**Proof:** by taking the least upper bound of the lower bounds. Converse also holds, dually.

**Example:** Every subset of the set of real numbers has a lub. This is an axiom of real numbers, the way they are defined (or constructed from rationals).

**Lemma:** In every lattice, $x \sqcup (x \sqcap y) = x$.  
**Proof:**   
We trivially have $x \sqsubseteq x \sqcup (x \sqcap y)$.  
Let’s prove that $x \sqcup (x \sqcap y) \sqsubseteq x$:  
$x$is an upper bound of $x$and $x \sqcap y$, $x \sqcup (x \sqcap y)$is the least upper bound of $x$and $x \sqcap y$, thus $x \sqcup (x \sqcap y) \sqsubseteq x $.

**Definition:** A lattice is *distributive* iff   
\begin{displaymath}
\begin{array}{l}
    x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \\
    x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)
\end{array}
\end{displaymath}

**Example:** Lattice of all subsets of a set is distributive. Linear order is a distributive lattice. See examples of non-distributive lattices in [Distributive lattice](http://www.google.com/search?q=Distributive%20lattice&btnI=lucky) and the characterization of non-distributive lattices.

**References**

* [Lattice (order)](http://en.wikipedia.org/wiki/Lattice%20%28order%29)
* [lecture notes by J.B. Nation](http://bigcheese.math.sc.edu/%7Emcnulty/alglatvar/lat0.pdf) or
* [Chapter I of a Course in Universal Algebra](http://bigcheese.math.sc.edu/%7Emcnulty/alglatvar/burrissanka.pdf).